CLOSURE OPERATORS AND PROJECTIONS ON INVOLUTION POSETS

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1. Introduction

Investigations of closure operators on an involution poset T lead to a certain type of closure operators (so called c-closure operators) that are closely related to projections on T.

In terms of these operators we give a necessary and sufficient condition for an involution poset to be an orthomodular lattice. An involution poset is an orthomodular lattice if and only if it admits certain c-closure operators. In that case, if L is an orthomodular lattice, the set of c-closure operators, under the usual ordering of closure operators, is orderisomorphic to the set of projections of the Baer *-semigroup $B(L)$ of hemimorphisms on L [4]. In this sense, but working on the "opposite end", this treatment enlarges that given in [3] where a similar necessary and sufficient condition is represented but for orthocomplemented posets and for mappings which in the case of an orthomodular lattice are exactly the closed projections of $B(L)$. C-closure operators appear as a natural generalization of symmetric closure operators [5].

2. C-closure operators

An *involution poset* T is a poset with largest element (1) and a mapping $e \in T \to e' \in T$ such that $e'' = e$ and $e \leqq f \Rightarrow f' \leqq e'$. For basic definitions see [1,2].

A *projection* ϕ on an involution poset T is a mapping $\phi \colon T \to T$ with the following properties:

i) $e \leqq f \Rightarrow e\phi \leqq f\phi$,

ii) $(e\phi)\phi = e\phi$,

iii) $(e\phi)'\phi \leqq e' \quad (e,f \in T)$.

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The set of projections on T, denoted by $P(T)$, is not empty since I defined by $eI := e$ is a projection.

LEMMA 1. *Let ϕ be a projection on T. Then $((e\phi)'\phi)'\phi = e\phi$ is valid for all $e \in T$.*

PROOF. Since $(e\phi)' \leqq e'$ for all $e \in T$, it follows that $((e\phi)'\phi)'\phi \leqq ((e\phi)')' = e\phi$. Clearly $e \leqq ((e\phi)'\phi)'$. Using monotony, we get from the latter inequality $e\phi \leqq ((e\phi)'\phi)'\phi$. Hence $e\phi = ((e\phi)'\phi)'\phi$.

REMARK 1. Let L be an orthomodular lattice. A projection $\phi \in P(L)$ is a join-homomorphism of L [2, Theorem 5.2, page 37]. On the other hand every join-homomorphism is monotone. From $(1\phi)'\phi \leqq 1'$ we get $0\phi = 0$, where $0 := 1'$. Therefore $P(L)$ coincides with the set of projections introduced by Foulis [4], namely the set of idempotent, self-adjoint hemimorphisms on L.

One verifies that in an involution poset T a closure operator γ satisfies

$$((e\gamma)'\gamma)'\gamma \leqq e\gamma \quad (e \in T).$$

Those closure operators for which the equality

$$((e\gamma)'\gamma)'\gamma = e\gamma \qquad (e \in T)$$

is valid are of special interest. As we will see below they are closely related to projections and determining for the lattice and orthomodular structure of T. We call these operators *c-closure operators* and denote with $C(T)$ the set of all c-closure operators on an involution poset T. The mappings I and $eJ := 1$ are c-closure operators.

$C(T)$ is partially ordered by means of the ordering relation

$$\gamma_1 \leqq \gamma_2 :\Leftrightarrow e\gamma_2 \leqq e\gamma_1 \qquad (e \in T).$$

I is the largest and J the smallest element of $C(T)$.

THEOREM 2. *Let T be an involution poset. If γ is a c-closure operator, then $((e\gamma)'\gamma)'$ is a projection on T. If ϕ is a projection, then $((e\phi)'\phi)'$ is a c-closure operator on T.*

The mapping $\gamma \in C(T) \to \phi \in P(T)$ where $e\phi := ((e\gamma)'\gamma)'$ is one-to-one and maps the set of c-closure operators onto the set of projections on T. $\phi \in P(T) \to \gamma \in C(T)$ where $e\gamma := ((e\phi)'\phi)'$ is the corresponding inverse mapping.

PROOF. Clearly the mapping $e \to ((e\gamma)'\gamma)'$ is monotone. Using properties of c-closure operators, we get

$$((((e\gamma)'\gamma)'\gamma)'\gamma)' = ((e\gamma)'\gamma)',$$

which proves idempotence of the mapping. Furthermore

$$(((((e\gamma)'\gamma)')'\gamma)'\gamma)' = (((e\gamma)'\gamma)'\gamma)' = (e\gamma)' \leqq e' .$$

Hence the mapping is a projection.

Let ϕ be a projection. By i) and iii) of the definition of a projection, one easily sees that the mapping $e\gamma: = ((e\phi)'\phi)'$ is monotone and majorizes the argument. By Lemma 1 and the basic properties of projections we get

$$(e\gamma)\gamma = ((((e\phi)'\phi)'\phi)'\phi)' = ((e\phi)'\phi)' = e\gamma$$

and

$$((e\gamma)'\gamma)'\gamma = ((((((e\phi)'\phi)\phi)'\phi)\phi)'\phi)' = ((((e\phi)'\phi)'\phi)'\phi)' = ((e\phi)'\phi)' = e\gamma .$$

Hence $\gamma \in C(T)$.

For all $\phi \in P(T)$, $\gamma \in C(T)$ and $e \in T$

$$((((e\gamma)'\gamma)''\gamma)'\gamma)'' = ((e\gamma)'\gamma)'\gamma = e\gamma$$

and

$$((((e\phi)'\phi)''\phi)'\phi)'' = ((e\phi)'\phi)'\phi = e\phi$$

is valid. This proves the second part of the theorem.

REMARK 2. Because of the one-to-one correspondence between $P(T)$ and $C(T)$ the ordering in the set of c-closure operators induces an ordering in the set of projections as follows:

Let ϕ_1, ϕ_2 be two projections and γ_1, γ_2 the corresponding c-closure operators. The relation

$$\phi_1 \leqq \phi_2 :\Leftrightarrow \gamma_1 \leqq \gamma_2$$

is an ordering relation that makes $P(T)$ into a partially ordered set. The mapping $\gamma \to \phi$ where $e\phi: = ((e\gamma)'\gamma)'$ can then be interpreted as an order-isomorphism between the posets $C(T)$ and $P(T)$.

The next two lemmata lead us to the main result of this paper.

LEMMA 3. *Let T be an orthocomplemented poset and $\gamma \in C(T)$. Then*
i) *$e\gamma \vee (e\gamma)'\gamma$ exists and is equal to 1,*
ii) *$e\gamma \wedge (e\gamma)'\gamma$ exists and is equal to 0γ.*

PROOF. i) Of course $e\gamma \leqq 1$ and $(e\gamma)'\gamma \leqq 1$. If there is an $f \in T$ such that $e\gamma \leqq f$ and $(e\gamma)'\gamma \leqq f$, then also $(e\gamma)' \leqq f$ since $(e\gamma)' \leqq (e\gamma)'\gamma$. But $e\gamma \vee (e\gamma)' = 1$, hence $1 \leqq f$. This proves that $e\gamma \vee (e\gamma)'\gamma = 1$. ii) By monotony $o\gamma \leqq e\gamma$ and $0\gamma \leqq (e\gamma)'\gamma$. Let $f \in T$ be an element such that $f \leqq e\gamma$ and $f \leqq (e\gamma)'\gamma$. By monotony and idempotence of the closure operator we get

$$f\gamma \leqq e\gamma \text{ and } f\gamma \leqq (e\gamma')\gamma \text{ or } (e\gamma)' \leqq (f\gamma)'$$

and $((e\gamma)'\gamma)' \leqq (f\gamma)'$. Again by monotony we have then $(e\gamma)'\gamma \leqq (f\gamma)'\gamma$ and

$$e\gamma = ((e\gamma)'\gamma)'\gamma \leqq (f\gamma)'\gamma.$$

According to part i) of this proof, this implies that $(f\gamma)'\gamma = 1$ or $((f\gamma)'\gamma)' = 0$. Finally we get $f \leqq f\gamma = ((f\gamma)'\gamma)'\gamma = 0\gamma$. Thus $e\gamma \wedge (e\gamma)'\gamma = 0\gamma$.

LEMMA 4. *Let T be an involution poset and γ a c-closure operator, then* $(0\gamma)'\gamma = 1$.

PROOF. By theorem 2 there is a projection ϕ such that $e\gamma = ((e\phi)'\phi)'$. Since $0\phi = 0$ and by lemma 1 we get

$$(0\gamma)'\gamma = ((((0\phi)'\phi)''\phi)'\phi)' = (((0\phi)'\phi)'\phi)' = (0\phi)' = 1.$$

THEOREM 5. *Let T be an involution poset. T is an orthomodular lattice if and only if every interval $[e,1]$ $(e \in T)$ is the range of a c-closure operator.*

PROOF. Assume that T is an orthomodular lattice. One verifies that for a given interval $[e,1]$ the mapping $f \to e \vee f$ is a closure operator that maps T onto it. We show that this mapping has the characteristic property of c-closure operators.

Since $e \leqq e \vee f$, there exists by orthomodularity of the lattice T an element $g \in T$ such that $e \vee g = e \vee f$ and $e \leqq g'$. Now

$$e \vee (e \vee (e \vee f)')' = e \vee (e \vee (e \vee g)')' = e \vee (e' \wedge (e \vee g)) = e \vee (e' \wedge g) = e \vee g = e \vee f.$$

Conversely, we prove first that T must be a lattice. When $e, f \in T$, then there is a c-closure operator γ that maps T onto the interval $[f,1]$. Clearly $e \leqq e\gamma$ and $f = 0\gamma \leqq e\gamma$. Let $g \in T$ be an element such that $e \leqq g$ and $f \leqq g$. Since γ maps T onto $[f,1]$, it follows fron the latter inequality that $g\gamma = g$. From $e \leqq g$ we then get $e\gamma \leqq g\gamma = g$. Thus $e \vee f$ exists in T and is equal to $e\gamma$.

Let $\gamma \in C(T)$ with $T\gamma = [e,1]$. By lemma 4 we get $1 = (0\gamma)'\gamma = e'\gamma = e' \vee e$ for all $e \in T$. Therefore T is an orthocomplemented lattice.

Now we prove orthomodularity of the lattice T. Let $e \leqq f$ and $\gamma \in C(T)$ such that $T\gamma = [e,1]$. We again have $e = 0\gamma$ and $f\gamma = f$. By Lemma 3 (ii) and the result above we get $e = 0\gamma = f\gamma \wedge (f\gamma)'\gamma = f \wedge f'\gamma = f \wedge (e \vee f')$.

REMARK 3. Let L be an orthomodular lattice. By Theorem 2 and Remark 1 the mappings $e \to e\phi := ((e\gamma)'\gamma)'$ $(\gamma \in C(L))$ are the projections in the Baer *-semigroup of hemimorphisms on L. One can prove that

$$(e\phi_1)\phi_2 = e\phi_1 \;\; (\phi_1, \phi_2 \in P(L);\, e \in L) \Leftrightarrow \phi_1 \leqq \phi_2,$$

thus the usual ordering of projections coincides with that induced by the poset $C(L)$ (Remark 2). The closed projections, namely the Sasaki-projections, are given by $((e\gamma_f)'\gamma_f)'$ $(f \in L)$ where $\gamma_f \in C(L)$ and $L\gamma_f = [f,1]$.

Note that a mapping γ is a symmetric closure operator on L [5] if and only if γ is a c-closure operator for which $0\gamma = 0$ is valid. Furthermore, the symmetric closure operators are the fixelements of the mappings exhibited in theorem 2.

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